5.5 (continued)

~~fa~~ from last time : $\vec{x}\,' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}$

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

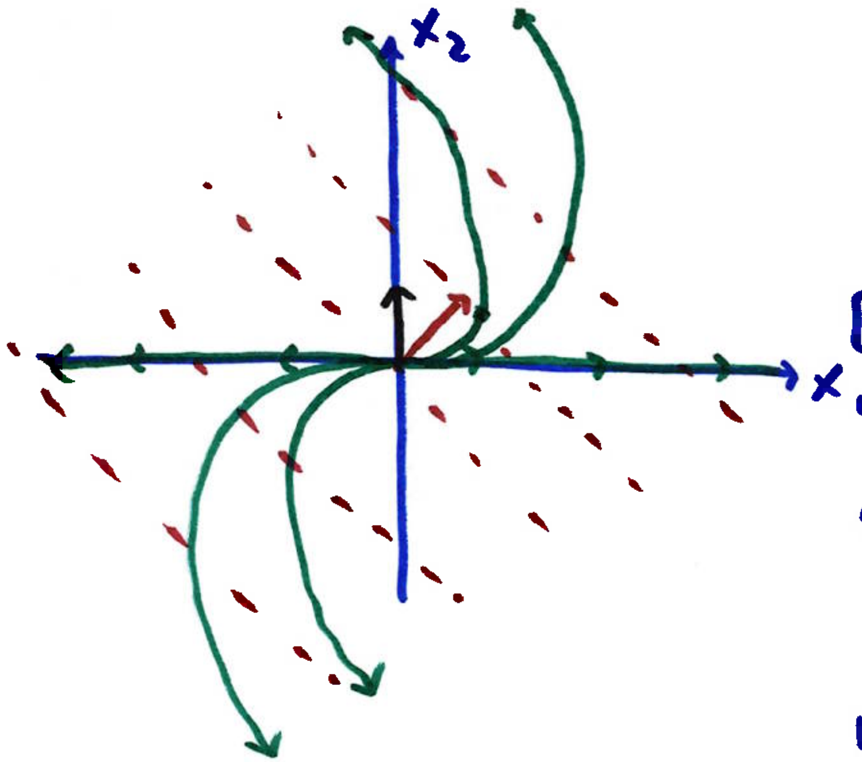
Sketch of the phase diagram

$\vec{x} \to \vec{0}$ as $t \to -\infty$ increasing t : move away

near the origin, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ dominates in $t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

leave origin along $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (true eigenvector)

as t increases, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ still dominates but $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ still contributes (both multiplied e^{2t})



$\begin{bmatrix}0\\1\end{bmatrix}$: generalized

red : sum of the two

$\begin{bmatrix}1\\0\end{bmatrix}$ (true eigenvector)

(above x_1 axis)

as t increases, we move left on the red dashed line and to the next "level"

below, as $t \to \infty$, move right

key point: only the true eigenvector is "visible"

(asymptote near origin)

now let's look at 3×3 systems (λ repeats 3 times)

possibilities:

1. full set of eigenvectors (matrix is complete)
2. missing one eigenvector (defect of one)
3. " two " (defect of two)

let's start with case 3 (missing two), the easier case

$\vec{x}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$ $\lambda = 2, 2, 2$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ missing two

find two generalized eigenvectors

let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

then, $(A - \lambda I)\vec{v}_2 = \vec{v}_1$ just like 2x2 systems

$(A - \lambda I)\vec{v}_3 = \vec{v}_2$

$(A - \lambda I)\vec{v}_2 = \vec{v}_1 \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

repeat for $\vec{v}_3$: $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

solution 1: $e^{\lambda t}\vec{v}_1$

solution 2: $e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

solution 3: $e^{\lambda t}\left(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)$

for this example, general solution is

$$\vec{x} = c_1 e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\right) + c_3 e^{2t}\left(\frac{1}{2}t^2\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

<u>alternative method</u> : from

$$(A-\lambda I)\vec{v}_1 = \vec{0}$$
$$(A-\lambda I)\vec{v}_2 = \vec{v}_1$$
$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$

$$(A-\lambda I)^2\vec{v}_3 = (A-\lambda I)\vec{v}_2 \qquad \text{mult. 3rd eq. by } (A-\lambda I)$$
$$= \vec{v}_1$$

one more time

$$(A-\lambda I)^3\vec{v}_3 = (A-\lambda I)\vec{v}_1 = \vec{0}$$

$$\boxed{\begin{array}{l}\text{missing } k \text{ vectors (defect of } k)\\ \text{then } (A-\lambda I)^{k+1} = \vec{0}\end{array}}$$

$$A-\lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (A-\lambda I)^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{(A-\lambda I)^{k+1} = 0 \leftarrow 0 \text{ matrix ALWAYS}}$$

revisit $\underbrace{(A-\lambda I)^3}_{0 \text{ matrix}} \vec{v}_3 = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{solve for } \vec{v}_3$$

$\vec{v}_3$ is arbitrary*

* pick any $\vec{v}_3 \neq \vec{0}$
and it must be
linearly indp from
true eigenvectors

true eigenvector: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

So, let's pick $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (but $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ e \\ \pi \end{bmatrix}$ are ok among others)

now rebuild the entire chain, including $\vec{v}_1$

$(A-\lambda I)\vec{v}_3 = \vec{v}_2$ ... $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$(A-\lambda I)\vec{v}_2 = \vec{v}_1$ ... $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

here, it happens to match the true eigenvector but it is NOT always the case

then solution 1: $e^{\lambda t}\vec{v}_1$

solution 2: $e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

" 3: $e^{\lambda t}(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3)$

I call the 1st method "stepping up"

" 2nd method "stepping down"

case of missing one is annoying w/ stepping up

$$\vec{x}' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \vec{x}$$

$\lambda = 1, 1, 1$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$ need $\vec{v}_3$

the defect is one so, $(A - \lambda I)^{1+1} \vec{v}_3 = \vec{0}$

$(A - \lambda I)^{1+1} = 0$ (0 matrix)

so, $\underbrace{(A - \lambda I)^2}_{\text{zeros}} \vec{v}_3 = \vec{0} \rightarrow \vec{v}_3$ is arbitrary

$\vec{v}_3 \neq 0$ AND indp from $\vec{v}_1$ and $\vec{v}_2$

$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

let's go w/ $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

rebuild the chain: $(A-\lambda I)\vec{v}_3 = \vec{v}_2$ ... $\vec{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}$

when we do $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$$\begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

not allowed

this means the $\vec{v}_2$ we found is in the eigenspace of the two true eigenvectors

pick either true vector to be $\vec{v}_1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

solutions: $e^{\lambda t}\vec{v}_1$

$e^{\lambda t}\vec{v}_2$

$e^{\lambda t}(t\vec{v}_2 + \vec{v}_3)$